

# Generating dyonic solutions in 5D Einstein-dilaton gravity with antisymmetric forms and dyonic black rings

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## Abstract

We consider 5-dimensional Einstein-dilaton gravity with antisymmetric forms. Assuming staticity and a restriction on the dilaton coupling parameters, we derive 4-dimensional sigma-model with a target space  $SL(2, R)/SO(1, 1) \times SL(2, R)/SO(1, 1)$ . On this basis, using the symmetries of the target space, we develop a solution generating technique and employ it to construct new asymptotically flat and non-flat dyonic black rings solutions. The solutions are analyzed and the basic physical quantities are calculated.

## 1 Introduction

An interesting development in the black holes studies is the discovery of the black ring solutions of the five-dimensional Einstein equations by Emparan and Reall [1], [2]. These are asymptotically flat solutions with an event horizon of topology  $S^2 \times S^1$  rather the much more familiar  $S^3$  topology. Moreover, it was shown in [2] that both the black hole and the the black ring can carry the same conserved charges, the mass and a single angular momentum, and therefore there is no uniqueness theorem in five dimensions. Since the Emparan and Reall's discovery many explicit examples of black ring solutions were found in various gravity theories [3]-[14]. Elvang was able to apply Hassan-Sen transformation to the solution [2] to find a charged black ring in the bosonic sector of the truncated heterotic string theory[3]. This solution is the first example of black rings with dipole charges depending, however, of the other physical parameters. A supersymmetric black ring in five-dimensional minimal supergravity was derived in [4] and then generalized to the case of concentric rings in [5] and [6]. A static black ring solution of the five dimensional Einstein-Maxwell gravity was found by Ida and Uchida in [15]. In [16] Emparan derived "dipole black rings" in Einstein-Maxwell-dilaton (EMd) theory in five dimensions. In this work Emparan showed that the black rings

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can exhibit novel feature with respect to the black holes. The black rings can also carry independent nonconserved charges which can be varied continuously without altering the conserved charges. This fact leads to continuous non-uniqueness. Following the same path that yields the three-charge rotating black holes [17]-[20], Elvang, Emparan and Figueras constructed a seven-parameter family of supergravity solutions that describe non-supersymmetric black rings and black tubes with three charges, three dipoles and two angular momenta [21].

The thermodynamics of the dipole black rings was studied first by Emparan in [16] and by Copsey and Horowitz in [22]. Within the framework of the quasilocal counterterm method, the thermodynamics of the dipole rings was discussed by Astefanesei and Radu [23]. The first law of black rings thermodynamics in  $n$ -dimensional Einstein dilaton gravity with  $(p+1)$ -form field strength was derived by Rogatko in [24]. Static and asymptotically flat black rings solutions in five-dimensional EMd gravity with arbitrary dilaton coupling parameter  $\alpha$  were presented in [25]. Non-asymptotically flat black rings immersed in external electromagnetic fields were found and discussed in [12], [25] and [26]. A systematical derivation of the asymptotically flat static black ring solutions in five-dimensional EMd gravity with an arbitrary dilaton coupling parameter was given in [27]. In the same paper and in [28], the author systematically derived new type static and rotating black ring solutions which are not asymptotically flat.

In the present work we consider 5-dimensional Einstein-dilaton gravity with antisymmetric forms. Assuming staticity we derive 4-dimensional sigma-model with a target space  $SL(2, R)/SO(1, 1) \times SL(2, R)/SO(1, 1)$ . On this basis, we develop a solution generating technique and employ it to construct new dyonic black ring solutions. The solutions are analyzed and the basic physical quantities are calculated.

## 2 Basic equations and solution generating

We consider the action

$$S = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left[ R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4} e^{-2\alpha\varphi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} e^{-2\beta\varphi} H^{\mu\nu\lambda} H_{\mu\nu\lambda} \right] \quad (1)$$

where  $H = dB$  and  $B$  is the Kalb–Ramond field. This action is the 5-dimensional version of the action studied by Gibbons and Maeda [29]. Let us note that the 3-form field strength  $H$  can be dualized to 2-form field strength  $\mathcal{F}$  whose contribution to the action is given by  $-\frac{1}{4} e^{2\beta\varphi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$ . In other words the theory we consider is equivalent to the Einstein-Maxwell-dilaton gravity with two distinct Maxwell fields and dilaton coupling parameters. Particular examples of the action (1) (or its dual version) with concrete values of the dilaton coupling parameters arise from string theory and supergravity via compactifications to five dimensions<sup>1</sup>.

The action (1) yields the following field equations

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<sup>1</sup>As a result of the compactifications, many additional fields come into play and one obtains rather complicated field models. In order to obtain a simplified (truncated) action, as one we consider here, we must suppress the additional fields by imposing certain selfconsistent conditions (see for example [30]- [32]).

$$\begin{aligned}
R_{\mu\nu} &= 2\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}e^{-2\alpha\varphi}\left(F_{\mu\lambda}F_\nu^\lambda - \frac{1}{6}F_{\sigma\lambda}F^{\sigma\lambda}g_{\mu\nu}\right) \\
&+ \frac{1}{4}e^{-2\beta\varphi}\left(H_{\mu\sigma\lambda}H_\nu^{\sigma\lambda} - \frac{2}{9}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}g_{\mu\nu}\right), \\
\nabla_\mu\nabla^\mu\varphi &= -\frac{\alpha}{8}e^{-2\alpha\varphi}F_{\sigma\lambda}F^{\sigma\lambda} - \frac{\beta}{24}e^{-2\beta\varphi}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}, \\
\nabla_\mu\left(e^{-2\alpha\varphi}F^{\mu\nu}\right) &= 0, \\
\nabla_\mu\left(e^{-2\beta\varphi}H^{\mu\nu\lambda}\right) &= 0.
\end{aligned} \tag{2}$$

Here we will consider static solutions. The metric of static spacetimes can be written in the form

$$ds^2 = -e^{2u}dt^2 + e^{-u}h_{ij}dx^i dx^j \tag{3}$$

where  $h_{ij}$  is four-dimensional metric with Euclidean signature. We take the fields  $F$  and  $H$  in the form

$$F = -2d\Phi \wedge dt, \tag{4}$$

$$H = -2e^{2\beta\varphi} \star (d\Psi \wedge dt). \tag{5}$$

Here  $\star$  is the Hodge dual and the potentials  $\Phi$  and  $\Psi$  depend on the coordinates  $x^i$  only. With all these assumptions the field equations are reduced to the following system

$$\mathcal{D}_i\mathcal{D}^iu = \frac{4}{3}e^{-2u-2\alpha\varphi}h^{ij}\partial_i\Phi\partial_j\Phi + \frac{4}{3}e^{-2u+2\beta\varphi}h^{ij}\partial_i\Psi\partial_j\Psi, \tag{6}$$

$$\mathcal{D}_i\mathcal{D}^i\varphi = \alpha e^{-2u-2\alpha\varphi}h^{ij}\partial_i\Phi\partial_j\Phi - \beta e^{-2u+2\beta\varphi}h^{ij}\partial_i\Psi\partial_j\Psi, \tag{7}$$

$$\mathcal{D}_i\left(e^{-2u-2\alpha\varphi}\mathcal{D}^i\Phi\right) = 0, \tag{8}$$

$$\mathcal{D}_i\left(e^{-2u+2\beta\varphi}\mathcal{D}^i\Psi\right) = 0, \tag{9}$$

$$R(h)_{ij} = \frac{3}{2}\partial_i u \partial_j u + 2\partial_i\varphi\partial_j\varphi - 2e^{-2u-2\alpha\varphi}\partial_i\Phi\partial_j\Phi - 2e^{-2u+2\beta\varphi}\partial_i\Psi\partial_j\Psi, \tag{10}$$

where  $\mathcal{D}_i$  and  $R(h)_{ij}$  are the covariant derivative and the Ricci tensor with respect to the metric  $h_{ij}$ .

These equations can be derived from the action

$$\begin{aligned}
S = \frac{1}{16\pi} \int d^4x \sqrt{h} [ &R(h) - \frac{3}{2}h^{ij}\partial_i u \partial_j u - 2h^{ij}\partial_i\varphi\partial_j\varphi \\
&+ 2e^{-2u-2\alpha\varphi}h^{ij}\partial_i\Phi\partial_j\Phi + 2e^{-2u+2\beta\varphi}h^{ij}\partial_i\Psi\partial_j\Psi ].
\end{aligned} \tag{11}$$

It is convenient to introduce the rescaled potentials and parameters :

$$\varphi_* = \frac{2}{\sqrt{3}}\varphi, \quad \Phi_* = \frac{2}{\sqrt{3}}\Phi, \quad \Psi_* = \frac{2}{\sqrt{3}}\Psi, \quad (12)$$

$$\alpha_* = \frac{\sqrt{3}}{2}\alpha, \quad \beta_* = \frac{\sqrt{3}}{2}\beta. \quad (13)$$

It turns out that the action (11) possesses an important group of symmetries when the coupling parameters  $\alpha$  and  $\beta$  satisfy<sup>2</sup>

$$\alpha_*\beta_* = 1. \quad (14)$$

In order to see that, we define the new fields  $\xi = u + \alpha_*\varphi_*$  and  $\eta = u - \beta_*\varphi_*$  and introduce the symmetric matrices

$$M_1 = e^{-\xi} \begin{pmatrix} e^{2\xi} - (1 + \alpha_*^2)\Phi_*^2 & -\sqrt{1 + \alpha_*^2}\Phi_* \\ -\sqrt{1 + \alpha_*^2}\Phi_* & -1 \end{pmatrix}, \quad (15)$$

$$M_2 = e^{-\eta} \begin{pmatrix} e^{2\eta} - (1 + \beta_*^2)\Psi_*^2 & -\sqrt{1 + \beta_*^2}\Psi_* \\ -\sqrt{1 + \beta_*^2}\Psi_* & -1 \end{pmatrix}, \quad (16)$$

with  $\det M_1 = \det M_2 = -1$  and  $\beta_* = 1/\alpha_*$ . Then the action (11) can be written into the form

$$S = \frac{1}{16\pi} \int d^4x \sqrt{h} \left[ R(h) + \frac{3}{4(1 + \alpha_*^2)} h^{ij} \text{Tr} \left( \partial_i M_1 \partial_j M_1^{-1} \right) + \frac{3\alpha_*^2}{4(1 + \alpha_*^2)} h^{ij} \text{Tr} \left( \partial_i M_2 \partial_j M_2^{-1} \right) \right]. \quad (17)$$

It is now clear that the action is invariant under the action of the group  $SL(2, R) \times SL(2, R)$

$$M_1 \rightarrow G_1 M_1 G_1^T, \quad M_2 \rightarrow G_2 M_2 G_2^T, \quad (18)$$

where  $G_1, G_2 \in SL(2, R)$ . The matrices  $M_1$  and  $M_2$  parameterize a coset  $SL(2, R)/SO(1, 1)$ . Therefore the action corresponds to a non-linear  $\sigma$ -model action with a target space  $SL(2, R)/SO(1, 1) \times SL(2, R)/SO(1, 1)$ . This is a generalization of  $\sigma$ -model studied in [27] and [28]. As one should expect, the coset space is a product of two identical cosets which is a consequence of the fact that our theory is equivalent to Einstein-Maxwell-dilaton gravity with two distinct Maxwell fields.

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<sup>2</sup>It has to be noted that this condition fixes values of  $\alpha$  (and  $\beta$ ) different from those predicted by string theory.

In the asymptotically flat case <sup>3</sup>, without loss of generality we can set

$$u(\infty) = \varphi(\infty) = \Phi(\infty) = \Psi(\infty) = 0, \quad (19)$$

i.e.

$$M_1(\infty) = M_2(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \quad (20)$$

The transformations preserving the asymptotics are those satisfying

$$B_{1,2} \sigma_3 B_{1,2}^T = \sigma_3. \quad (21)$$

Therefore we find that  $B_{1,2} \in SO(1,1) \subset SL(2,R)$ . Here we will adopt the parameterization

$$B_{1,2} = \begin{pmatrix} \cosh \theta_{1,2} & \sinh \theta_{1,2} \\ \sinh \theta_{1,2} & \cosh \theta_{1,2} \end{pmatrix}. \quad (22)$$

The symmetries of the dimensionally reduced field equations (6) can be employed to generate new solutions from known ones, in particular, new solutions from known solutions of the vacuum Einstein equations. Let us consider a static solution of the five-dimensional vacuum Einstein equations

$$ds_0^2 = -e^{2u_0} dt^2 + e^{-u_0} h_{ij} dx^i dx^j \quad (23)$$

which is encoded into the matrices

$$M_1^{(0)} = M_2^{(0)} = \begin{pmatrix} e^{u_0} & 0 \\ 0 & -e^{-u_0} \end{pmatrix} \quad (24)$$

and the metric  $h_{ij}$ . The  $SO(1,1) \times SO(1,1)$  transformations then generate new static solutions to the field equations (2) given by the matrices

$$M_1 = B_1 M_1^{(0)} B_1^T, \quad M_2 = B_2 M_2^{(0)} B_2^T \quad (25)$$

and the same metric  $h_{ij}$ . In explicit form we have

$$e^{2u} = e^{2u_0} \frac{\left( \cosh^2 \theta_1 - e^{2u_0} \sinh^2 \theta_1 \right)^{-2/(1+\alpha_*^2)}}{\left( \cosh^2 \theta_2 - e^{2u_0} \sinh^2 \theta_2 \right)^{2\alpha_*^2/(1+\alpha_*^2)}},$$

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<sup>3</sup>More precisely, we mean solutions with "flat asymptotic" for the matrices  $M_1$  and  $M_2$ , i.e.  $M_1(\infty) = M_2(\infty) = \sigma_3$ .

$$\begin{aligned}
e^{-2\alpha_*\varphi_*} &= \left[ \frac{\cosh^2 \theta_1 - e^{2u_0} \sinh^2 \theta_1}{\cosh^2 \theta_2 - e^{2u_0} \sinh^2 \theta_2} \right]^{2\alpha_*^2/(1+\alpha_*^2)} \\
\Phi_* &= \frac{\tanh \theta_1}{\sqrt{1+\alpha_*^2}} \frac{1 - e^{2u_0}}{1 - e^{2u_0} \tanh^2 \theta_1}, \\
\Psi_* &= \frac{\alpha_* \tanh \theta_2}{\sqrt{1+\alpha_*^2}} \frac{1 - e^{2u_0}}{1 - e^{2u_0} \tanh^2 \theta_2}.
\end{aligned} \tag{26}$$

In the next section we shall employ these solution generating formulas in order to obtain new black ring solutions to the field equations (2).

### 3 Dyonic black rings

As an explicit example of a seed vacuum Einstein solution we consider the five-dimensional, static black ring solution given by the metric

$$\begin{aligned}
ds_0^2 &= -\frac{F(y)}{F(x)} dt^2 \\
&+ \frac{\mathcal{R}^2}{(x-y)^2} \left[ F(x)(y^2-1)d\psi^2 + \frac{F(x)}{F(y)} \frac{dy^2}{(y^2-1)} + \frac{dx^2}{(1-x^2)} + F(x)(1-x^2)d\phi^2 \right]
\end{aligned} \tag{27}$$

where  $F(x) = 1 + \lambda x$ ,  $\mathcal{R} > 0$  and  $0 < \lambda < 1$ . The coordinate  $x$  is in the range  $-1 \leq x \leq 1$  and the coordinate  $y$  is in the range  $-\infty < y \leq -1$ . The solution has a horizon at  $y = -1/\lambda$ . The topology of the horizon is  $S^2 \times S^1$ , parameterized by  $(x, \phi)$  and  $\psi$ , respectively. In order to avoid a conical singularity at  $y = -1$  one must demand that the period of  $\psi$  satisfies  $\Delta\psi = 2\pi/\sqrt{1-\lambda}$ . If one demands regularity at  $x = -1$  the period of  $\phi$  must be  $\Delta\phi = 2\pi/\sqrt{1-\lambda}$ . In this case the solution is asymptotically flat and the ring is sitting on the rim of disk shaped membrane with a negative deficit angle. To enforce regularity at  $x = 1$  one must take  $\Delta\phi = 2\pi/\sqrt{1+\lambda}$  and the solution describes a black ring sitting on the rim of a disk shaped hole in an infinitely extended deficit membrane with positive deficit. More detailed analysis of the black ring solution can be found in [1].

Applying the transformation (26) to the neutral black ring solution (27) we obtain the following dyonic solution

$$\begin{aligned}
ds^2 &= -\frac{F(y)}{F(x)} \frac{\left( \cosh^2 \theta_1 - \frac{F(y)}{F(x)} \sinh^2 \theta_1 \right)^{-2/(1+\alpha_*^2)}}{\left( \cosh^2 \theta_2 - \frac{F(y)}{F(x)} \sinh^2 \theta_2 \right)^{2\alpha_*^2/(1+\alpha_*^2)}} dt^2 \\
&+ \frac{\left( \cosh^2 \theta_1 - \frac{F(y)}{F(x)} \sinh^2 \theta_1 \right)^{1/(1+\alpha_*^2)}}{\left( \cosh^2 \theta_2 - \frac{F(y)}{F(x)} \sinh^2 \theta_2 \right)^{-\alpha_*^2/(1+\alpha_*^2)}} \times \\
&\frac{\mathcal{R}^2}{(x-y)^2} \left[ F(x)(y^2-1)d\psi^2 + \frac{F(x)}{F(y)} \frac{dy^2}{(y^2-1)} + \frac{dx^2}{(1-x^2)} + F(x)(1-x^2)d\phi^2 \right],
\end{aligned} \tag{28}$$

$$e^{-2\alpha\varphi} = \left[ \frac{\cosh^2 \theta_1 - \frac{F(y)}{F(x)} \sinh^2 \theta_1}{\cosh^2 \theta_2 - \frac{F(y)}{F(x)} \sinh^2 \theta_2} \right]^{2\alpha_*^2/(1+\alpha_*^2)}, \quad (29)$$

$$\Phi = \frac{\sqrt{3} \tanh \theta_1}{2\sqrt{1+\alpha_*^2}} \frac{1 - \frac{F(y)}{F(x)}}{1 - \frac{F(y)}{F(x)} \tanh^2 \theta_1}, \quad (30)$$

$$\Psi = \frac{\sqrt{3}\alpha_* \tanh \theta_2}{2\sqrt{1+\alpha_*^2}} \frac{1 - \frac{F(y)}{F(x)}}{1 - \frac{F(y)}{F(x)} \tanh^2 \theta_2}. \quad (31)$$

It is worth noting that this new solution is a generalization of the solution of [25] which can be obtained as a particular case by setting  $\theta_2 = 0$ . The analysis of our dyonic solution is quite similar to that for the neutral black ring [1]. From the explicit form of the metric it is clear that there is a horizon at  $y = -1/\lambda$ . Using arguments similar to those in [1] one can show that the horizon is regular. The metric of a constant  $t$  slice through the horizon is

$$ds_h^2 = [\cosh^2 \theta_1]^{1/(1+\alpha_*^2)} [\cosh^2 \theta_2]^{\alpha_*^2/(1+\alpha_*^2)} \frac{\mathcal{R}^2 \lambda^2}{F^2(x)} \times \left[ F(x) \frac{1-\lambda^2}{\lambda^2} d\psi^2 + \frac{dx^2}{1-x^2} + F(x)(1-x^2) d\phi^2 \right]. \quad (32)$$

To analyze the case when  $y \rightarrow -1$ , we set  $y = -\cosh \rho$ . Near  $\rho = 0$  the  $ty$  part of the metric is conformal to

$$dl_{ty}^2 \approx d\rho^2 + (1-\lambda)\rho^2 d\psi^2. \quad (33)$$

This is regular at  $\rho = 0$  provided that  $\psi$  is identified with period  $\Delta\psi = 2\pi/\sqrt{1-\lambda}$ . Let us consider now the  $x\phi$  part of the metric which is conformal to

$$dl_{x\phi}^2 \approx \frac{dx^2}{1-x^2} + F(x)(1-x^2) d\phi^2. \quad (34)$$

In order for this metric to be regular at  $x = -1$  we must identify  $\phi$  with period  $\Delta\phi = 2\pi/\sqrt{1-\lambda}$ . The regularity at  $x = 1$  requires to impose the period  $\Delta = 2\pi/\sqrt{1+\lambda}$ . It is therefore not possible to have regularity at both  $x = -1$  and  $x = 1$ . The regularity at  $x = -1$  means the presence of a conical singularity at  $x = 1$  and vice versa. In both cases the  $x\phi$  part of the metric describes a surface that is topologically  $S^2$  with a conical singularities at one of the poles. The above analysis show that the horizon metric (32) describes a hypersurface with topology  $S^2 \times S^1$ . In addition, there is an inner spacelike singularity at  $y = -\infty$ .

Using the explicit form of the horizon metric (32) one can calculate the area of the horizon

$$\mathcal{A}_h^\pm = \frac{8\pi^2 \lambda^2 \mathcal{R}^3}{(1-\lambda)\sqrt{(1+\lambda)(1\pm\lambda)}} [\cosh^3 \theta_1]^{1/(1+\alpha_*^2)} [\cosh^3 \theta_2]^{\alpha_*^2/(1+\alpha_*^2)}. \quad (35)$$

Here the sign  $+$  corresponds to the case of a conical singularity at  $x = -1$  and vice versa.

The temperature can be found by Euclideanizing the metric and the result is

$$T = \frac{\sqrt{1-\lambda^2}}{4\pi\mathcal{R}\lambda} [\cosh^3 \theta_1]^{-1/(1+\alpha_*^2)} [\cosh^3 \theta_2]^{-\alpha_*^2/(1+\alpha_*^2)}. \quad (36)$$

In order to compute the mass of the dyonic solution we use the quasilocal formalism. After a long algebra we find

$$M^\pm = \frac{3\pi\lambda\mathcal{R}^2}{4\sqrt{(1-\lambda)(1\pm\lambda)}} \left[ 1 + \frac{2}{1+\alpha_*^2} \sinh^2 \theta_1 + \frac{2\alpha_*^2}{1+\alpha_*^2} \sinh^2 \theta_2 \right]. \quad (37)$$

The electric and the magnetic charge are defined by

$$Q = \frac{1}{8\pi} \oint_{S_\infty^3} \star e^{-2\alpha\varphi} F, \quad (38)$$

$$P = \frac{1}{8\pi} \oint_{S_\infty^3} H. \quad (39)$$

After some algebra we find

$$Q^\pm = \frac{\sqrt{3} \sinh \theta_1 \cosh \theta_1}{\sqrt{1+\alpha_*^2}} \frac{\pi\lambda\mathcal{R}^2}{\sqrt{(1-\lambda)(1\pm\lambda)}}, \quad (40)$$

$$P^\pm = \frac{\sqrt{3}\alpha_* \sinh \theta_2 \cosh \theta_2}{\sqrt{1+\alpha_*^2}} \frac{\pi\lambda\mathcal{R}^2}{\sqrt{(1-\lambda)(1\pm\lambda)}}. \quad (41)$$

It is straightforward to check that the following Smarr-like relation is satisfied

$$M^\pm = \frac{3}{8} T \mathcal{A}_h^\pm + \Phi_h Q^\pm + \Psi_h P^\pm \quad (42)$$

where subscript  $h$  shows that the corresponding quantity is evaluated on the horizon.

## 4 Dyonic black hole limit

Here we consider the asymptotically flat dyonic black ring solution for which the conical singularity is at  $x = 1$ . Let us introduce the new parameter

$$m = \frac{2\mathcal{R}^2}{1-\lambda} \quad (43)$$

such that it remains finite as  $\lambda \rightarrow 1$  and  $\mathcal{R} \rightarrow 0$ . Also, let us change the coordinates

$$\begin{aligned} x &= -1 + \frac{2\mathcal{R}^2 \sin^2 \theta}{r^2 - m \sin^2 \theta}, \\ y &= -1 - \frac{2\mathcal{R}^2 \cos^2 \theta}{r^2 - m \sin^2 \theta}, \end{aligned} \quad (44)$$



and rescale  $\psi$  and  $\phi$

$$(\psi, \phi) \rightarrow \sqrt{\frac{m}{2\mathcal{R}^2}}(\psi, \phi) \quad (45)$$

so that they now have canonical periodicity  $2\pi$ . Then we obtain the solution

$$\begin{aligned} ds^2 &= -\left(1 - \frac{m}{r^2}\right) \frac{\left(1 + \frac{m}{r^2} \sinh^2 \theta_1\right)^{-2/(1+\alpha_*^2)}}{\left(1 + \frac{m}{r^2} \sinh^2 \theta_2\right)^{2\alpha_*^2/(1+\alpha_*^2)}} dt^2 \\ &\quad + \frac{\left(1 + \frac{m}{r^2} \sinh^2 \theta_1\right)^{1/(1+\alpha_*^2)}}{\left(1 + \frac{m}{r^2} \sinh^2 \theta_2\right)^{-\alpha_*^2/(1+\alpha_*^2)}} \left[ \frac{dr^2}{1 - \frac{m}{r^2}} + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 + r^2 \sin^2 \theta d\phi^2 \right], \\ e^{-2\alpha\varphi} &= \frac{\left[1 + \frac{m}{r^2} \sinh^2 \theta_1\right]^{2\alpha_*^2/(1+\alpha_*^2)}}{\left[1 + \frac{m}{r^2} \sinh^2 \theta_2\right]}, \\ \Phi &= \frac{\sqrt{3} \sinh \theta_1 \cosh \theta_1}{2\sqrt{1 + \alpha_*^2}} \frac{m}{r^2 + m \sinh^2 \theta_1}, \\ \Psi &= \frac{\sqrt{3} \alpha_* \sinh \theta_2 \cosh \theta_2}{2\sqrt{1 + \alpha_*^2}} \frac{m}{r^2 + m \sinh^2 \theta_2}. \end{aligned} \quad (46)$$

This solution describes a 5-dimensional asymptotically flat dyonic black hole with a horizon at  $r = m$  with usual  $S^3$ -topology. There is also a regular inner horizon at  $r = 0$  with the same topology. It is worth noting that the solution (46) can be obtained via the above discussed solution generating technique from the 5-dimensional Schwarzschild-Tangherlini black hole solution

$$ds_0^2 = -\left(1 - \frac{m}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{m}{r^2}} + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 + r^2 \sin^2 \theta d\phi^2 \quad (47)$$

with mass  $M = \frac{3\pi}{8}m$ . Our dyonic black hole solution is in fact the 5-dimensional Gibbons-Maeda dyonic black hole<sup>4</sup> derived via a completely different method in [29].

The physical quantities characterizing the dyonic black hole solution can be found as a limit of the corresponding quantities for the black ring. Also, it can be checked that a Smarr-like relation is satisfied.

## 5 Asymptotically non-flat dyonic black rings

In this section we consider more exotic black ring solutions - black rings with unusual asymptotic. These are dyonic black rings which are asymptotically non-flat. Such black hole and black brane solutions, although unusual, have attracted much interest [33]-[42].

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<sup>4</sup>Note that the 5D Gibbons-Maeda solution is written here in coordinates different from those used in [29].

In order to generate asymptotically non-flat dyonic black ring solutions we consider the special  $SL(2, R) \times SL(2, R)$  transformation<sup>5</sup> given by  $N \times B$  where

$$N = \begin{pmatrix} 0 & -a^{-1} \\ a & a \end{pmatrix}, \quad (48)$$

$$B = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}. \quad (49)$$

This transformation, applied to the seed solution (23), generates new dyonic solutions with the following potentials

$$e^{2u} = e^{2u_0} \frac{(\cosh^2 \theta - e^{2u_0} \sinh^2 \theta)^{-2\alpha_*^2/(1+\alpha_*^2)}}{[a^2(1 - e^{2u_0})]^{2/(1+\alpha_*^2)}}, \quad (50)$$

$$e^{-2\alpha_* \varphi_*} = \left[ \frac{a^2(1 - e^{2u_0})}{\cosh^2 \theta - e^{2u_0} \sinh^2 \theta} \right]^{2\alpha_*^2/(1+\alpha_*^2)}, \quad (51)$$

$$\Phi_* = \frac{1}{\sqrt{1 + \alpha_*^2}} \frac{a^{-2}}{1 - e^{2u_0}}, \quad (52)$$

$$\Psi_* = \frac{\alpha_* \tanh \theta}{\sqrt{1 + \alpha_*^2}} \frac{1 - e^{2u_0}}{1 - e^{2u_0} \tanh^2 \theta}. \quad (53)$$

Applying these formulas to the neutral black ring solution we obtain the following dyonic solution

$$\begin{aligned} ds^2 = & -\frac{F(y)}{F(x)} \frac{[\cosh^2 \theta - \frac{F(y)}{F(x)} \sinh^2 \theta]^{-2\alpha_*^2/(1+\alpha_*^2)}}{[a^2(1 - \frac{F(y)}{F(x)})]^{2/(1+\alpha_*^2)}} dt^2 \\ & + \left[ \cosh^2 \theta - \frac{F(y)}{F(x)} \sinh^2 \theta \right]^{\alpha_*^2/(1+\alpha_*^2)} \left[ a^2(1 - \frac{F(y)}{F(x)}) \right]^{1/(1+\alpha_*^2)} \times \\ & \frac{\mathcal{R}^2}{(x-y)^2} \left[ F(x)(y^2 - 1)d\psi^2 + \frac{F(x)}{F(y)} \frac{dy^2}{(y^2 - 1)} + \frac{dx^2}{(1 - x^2)} + F(x)(1 - x^2)d\phi^2 \right], \end{aligned} \quad (54)$$

$$e^{-2\alpha \varphi} = \left[ \frac{a^2(1 - \frac{F(y)}{F(x)})}{\cosh^2 \theta - \frac{F(y)}{F(x)} \sinh^2 \theta} \right]^{2\alpha_*^2/(1+\alpha_*^2)}, \quad (55)$$

$$\Phi = \frac{\sqrt{3}}{2} \frac{a^{-2}}{\sqrt{1 + \alpha_*^2}} \frac{1}{1 - \frac{F(y)}{F(x)}}, \quad (56)$$

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<sup>5</sup>The dual transformation  $B \times N$  generate a dual solution which can be also obtained via the "discrete duality"  $\varphi \rightarrow -\varphi$ ,  $\Phi \rightarrow \Psi$  and  $\alpha \rightarrow \beta$ . In addition, we can consider the transformation  $N(a) \times N(b)$ , too. This transformation, however, generates configurations with frozen (constant) dilaton field. Such configurations will not be considered here.

$$\Psi = \frac{\sqrt{3}\alpha_* \tanh \theta}{2} \frac{1 - \frac{F(y)}{F(x)}}{\sqrt{1 + \alpha_*^2} 1 - \frac{F(y)}{F(x)} \tanh^2 \theta} \quad (57)$$

The metric has a horizon at  $y = -1/\lambda$ . Arguments similar to those of [1] show that the horizon is regular. Further analysis of the solution is quite similar to the asymptotically flat case. That is why we present only the results. In order to get rid of the possible conical singularity at  $y = -1$  we must identify  $\psi$  with a period  $\Delta = 2\pi/\sqrt{1 - \lambda}$ . It is not possible to have regularity at  $x = 1$  and  $x = -1$ . The regularity at  $x = 1$  requires the period  $\Delta\phi = 2\pi/\sqrt{1 + \lambda}$  while to avoid the conical singularity at  $x = -1$  we must impose  $\Delta\phi = 2\pi/\sqrt{1 - \lambda}$ . Also, the solution has an inner spacelike singularity at  $y = -\infty$ .

In order to see the asymptotic behavior of the solution at spacial infinity ( $x = y = -1$ ) let us introduce the new coordinates

$$r \cos \theta = \mathcal{R} \frac{\sqrt{y^2 - 1}}{x - y}, \quad r \sin \theta = \mathcal{R} \frac{\sqrt{1 - x^2}}{x - y}, \quad \tilde{\psi} = \sqrt{1 - \lambda} \psi, \quad \tilde{\phi} = \sqrt{1 - \lambda} \phi. \quad (58)$$

Then for  $r \rightarrow \infty$  we find

$$ds^2 \approx - \left[ \frac{(1 - \lambda)}{2\lambda} \frac{r^2}{a^2 \mathcal{R}^2} \right]^{\frac{2}{1 + \alpha_*^2}} dt^2 \quad (59)$$

$$+ \left[ \frac{(1 - \lambda)}{2\lambda} \frac{r^2}{a^2 \mathcal{R}^2} \right]^{\frac{-1}{1 + \alpha_*^2}} \left[ dr^2 + r^2 d\theta^2 + r^2 \cos^2 \theta d\tilde{\psi}^2 + r^2 \sin^2 \theta d\tilde{\phi}^2 \right],$$

$$e^{2\alpha\varphi} \approx \left[ \frac{(1 - \lambda)}{2\lambda} \frac{r^2}{a^2 \mathcal{R}^2} \right]^{\frac{2\alpha_*^2}{1 + \alpha_*^2}}, \quad (60)$$

$$\Phi \approx \frac{\sqrt{3}}{2\sqrt{1 + \alpha_*^2}} \left[ \frac{(1 - \lambda)}{2\lambda} \frac{r^2}{a^2 \mathcal{R}^2} \right], \quad (61)$$

$$\Psi \approx \frac{\sqrt{3}\alpha_* \sinh \theta \cosh \theta}{2\sqrt{1 + \alpha_*^2}} \frac{2\lambda}{1 - \lambda} \frac{\mathcal{R}^2}{r^2}. \quad (62)$$

Even though our solution is not asymptotically flat and the dilaton field behaves like  $\varphi \sim \ln(r)$  for large  $r$ , the Ricci and the Kretschmann scalars vanish as  $r \rightarrow \infty$ . More precisely we have

$$R \sim r^{-\frac{2\alpha_*^2}{1 + \alpha_*^2}}, \quad R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \sim r^{-\frac{4\alpha_*^2}{1 + \alpha_*^2}}. \quad (63)$$

Therefore the spacetime is well-behaved for  $r \rightarrow \infty$ .

The temperature can be found via the surface gravity and the result is

$$T = \frac{\sqrt{1 - \lambda^2}}{4\pi \mathcal{R} \lambda} [\cosh \theta]^{-3\alpha_*^2/(1 + \alpha_*^2)} a^{-3/(1 + \alpha_*^2)}. \quad (64)$$

The horizon area is found by straightforward integration

$$\mathcal{A}_h^\pm = \frac{8\pi^2 \mathcal{R}^3 \lambda^2}{(1-\lambda)\sqrt{(1+\lambda)(1\pm\lambda)}} [\cosh \theta]^{3\alpha_*^2/(1+\alpha_*^2)} a^{3/(1+\alpha_*^2)}. \quad (65)$$

For the electric and the magnetic charge we obtain

$$Q^\pm = \frac{\sqrt{3}a^2}{\sqrt{1+\alpha_*^2}} \frac{\pi \mathcal{R}^2 \lambda}{\sqrt{(1-\lambda)(1\pm\lambda)}}, \quad (66)$$

$$P^\pm = \frac{\sqrt{3}\alpha_* \sinh \theta \cosh \theta}{\sqrt{1+\alpha_*^2}} \frac{\pi \mathcal{R}^2 \lambda}{\sqrt{(1-\lambda)(1\pm\lambda)}}. \quad (67)$$

The mass of the solution is calculated through the use of the quasilocal formalism. After a long algebra we find

$$M^\pm = \frac{3\pi \mathcal{R}^2 \lambda}{4\sqrt{(1-\lambda)(1\pm\lambda)}} \left[ \frac{\alpha_*^2}{1+\alpha_*^2} + \frac{2\alpha_*^2}{1+\alpha_*^2} \sinh^2 \theta \right]. \quad (68)$$

The electric potential  $\Phi$  is defined up to an arbitrary additive constant. In the asymptotically flat case there is a preferred gauge in which  $\Phi(\infty) = 0$ . In the non-asymptotically flat case the electric potential diverges at spacial infinity and there is no preferred gauge. The arbitrary constant, however, can be fixed so that the Smarr-type relation is satisfied:

$$M^\pm = \frac{3}{8} T A_h^\pm + \Phi_h Q^\pm + \Psi_h P^\pm. \quad (69)$$

## 6 Asymptotically non-flat dyonic black hole limit

Here we consider the dyonic black ring solution with a conical singularity at  $x = 1$ . Performing the coordinate change (44), the coordinate rescaling (45) and taking the limit  $\mathcal{R} \rightarrow 0$  and  $\lambda \rightarrow 1$  with  $m = 2\mathcal{R}^2/(1-\lambda)$  fixed, we obtain the following asymptotically non-flat dyonic black hole solution

$$\begin{aligned} ds^2 &= - \left(1 - \frac{m}{r^2}\right) \left(\frac{r^2}{ma^2}\right)^{2/(1+\alpha_*^2)} \left(1 + \frac{m}{r^2} \sinh^2 \theta\right)^{-2\alpha_*^2/(1+\alpha_*^2)} dt^2 \\ &+ \left(1 + \frac{m}{r^2} \sinh^2 \theta\right)^{\alpha_*^2/(1+\alpha_*^2)} \left(\frac{r^2}{ma^2}\right)^{-1/(1+\alpha_*^2)} \times \\ &\quad \left[ \frac{dr^2}{1 - \frac{m}{r^2}} + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 + r^2 \sin^2 \theta d\phi^2 \right], \\ e^{-2\alpha\varphi} &= \left[ \frac{ma^2}{r^2 + m \sinh^2 \theta} \right]^{2\alpha_*^2/(1+\alpha_*^2)}, \end{aligned} \quad (70)$$

$$\begin{aligned}\Phi &= \frac{\sqrt{3}}{2\sqrt{1+\alpha_*^2}} \frac{r^2}{ma^2}, \\ \Psi &= \frac{\sqrt{3}\alpha_* \sinh \theta \cosh \theta}{2\sqrt{1+\alpha_*^2}} \frac{m}{r^2 + m \sinh^2 \theta}.\end{aligned}$$

This solution describes a 5-dimensional asymptotically non-flat dyonic black hole with a horizon at  $r = m$  with usual  $S^3$ -topology. There is also a regular inner horizon at  $r = 0$  with the same topology. It is worth noting that the solution (70) can be obtained via the above discussed solution generating technique from the 5-dimensional Schwarzschild-Tangherlini black hole solution (47). The physical quantities characterizing the dyonic black hole solution can be found as a limit of the corresponding quantities for the black ring. In addition, it can be checked, that the mass, the charges and the horizon potentials satisfy a Smarr-like relation.

## 7 Conclusion

In this work we considered static 5-dimensional Einstein-dilaton gravity with antisymmetric forms. We derived a 4-dimensional  $\sigma$ -model with a target space  $SL(2, R)/SO(1, 1) \times SL(2, R)/SO(1, 1)$ . Using the target space symmetries we constructed new asymptotically flat and non-flat dyonic black rings solutions. These solutions were analyzed and the basic physical quantities were calculated. It was shown that Smarr-type relations are satisfied.

The solution generating technique presented in this paper can be used for generating not only black ring and black hole solutions but for generating many others exact solutions. It is worth noting that the formulation of the static equations in terms of a non-linear  $\sigma$ -model enables us to apply the integrable system techniques when the target space variables depend on two coordinates only (i.e. in the presence of two additional commuting Killing vectors). In this connection we should mention the related works on soliton solutions in 5-dimensional general relativity [13],[43],[44] and in some 5-dimensional low-energy string theories [45] - [47].

Finally, it would be interesting to find rotating dyonic black rings. The construction of dyonic solutions with rotation is now in progress and the results will be presented elsewhere.

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